THE CLOSING LEMMA FOR GENERALIZED RECURRENCE IN THE PLANE

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ABSTRACT. We prove a version of the Closing Lemma for C^r vector fields on the plane, $r \geq 1$, and for a kind of recurrence obtained using the concept of prolongational limit sets. We call it generalized recurrence.

Given a nonperiodic point p in the generalized recurrent set we perturb the vector field in order to get a new vector field arbitrarily close to it, with a closed orbit through p.

1. Introduction. This paper deals with the C^r Closing Lemma, $r \geq 1$ on a noncompact manifold, the plane $\mathbf{R}^2$. Before we state precisely the hypothesis of the main theorem, we want to make some brief historical comments.

R. Thom, in 1960, was the first to consider the following problem. Can a vector field with a recurrent trajectory through a point p be perturbed so as to obtain a new vector field with a closed trajectory through p? He claimed an affirmative answer, but his argument was valid only for a C^0-small perturbation. The perception that this problem is trivial in class C^0 and very difficult, $r \geq 1$, is due to M. M. Peixoto in [12].

In 1965, Charles Pugh proved the C^1 Closing Lemma for compact manifolds of dimension two and three. In 1967 he generalized it for arbitrary dimension and extended it to the case of closing a nonwandering trajectory, rather than a recurrent one [16]. In 1973, he proved that for a weaker type of recurrent point, for which $\alpha(p) \cap \omega(p) \neq \varnothing$, the C^2-double closing is not always possible on the 2-torus T^2 [17]. In 1983, C. Pugh and C. Robinson [18] established the Closing Lemma when M is noncompact, provided the point p lies on $\Omega_c = \{p \in \Omega: \alpha(p) \cup \omega(p) \neq \varnothing\}$.

It should be remarked that the C^r Closing Lemma, $r \geq 1$, in the case that M is the 2-torus T^2 and the vector field never vanishes, was proved in 1962 by M. M. Peixoto [12] and recently by C. Gutierrez in [4] for the so called constant type of vector fields on T^2 with finitely many singularities. In [5] C. Gutierrez gives a counterexample to the C^2 Closing Lemma for the punctured torus. There is also "An Ergodic Closing Lemma" of R. Mañe (1982) that was used to characterize structurally stable diffeomorphisms of two-manifolds [6].

In the plane we do not have nontrivial recurrence, but there is a subtle kind of recurrence, the prolongational recurrence. So it makes sense to deal with the C^r Closing Lemma in $\mathbf{R}^2$. A recent counterexample of C. Pugh in the punctured torus [19] disproves the C^1 Closing Lemma for prolongational recurrence on noncompact two-manifolds in general, but on compact ones it is an open question.

In this paper, we consider the C^r vector fields X on $\mathbf{R}^2$, $r \geq 1$, which generate a flow φ, i.e., the trajectory through every point is defined for all time. The space $\mathscr{X}^r(\mathbf{R}^2)$ of these vector fields is endowed with the C^r Whitney topology. That is the appropriate topology for the noncompact case (see [13 and 14]). In order to state the C^r Closing Lemma on $\mathbf{R}^2$, we give some basic definitions.

(1.1) DEFINITION. For each $X \in \mathscr{X}^r(\mathbf{R}^2)$ and $x \in \mathbf{R}^2$, define the *first positive* (resp. negative) *prolongational limit set* by

$$J^{\overset{+}{(-)}}(x) = \{y \in \mathbf{R}^2 : \exists x_n \to x, t_n \to \infty \text{ (resp. } t_n \to -\infty) \text{ s.t. } \varphi_{t_n}(x_n) \to y\}.$$

(1.2) DEFINITION. For a subset $S \subset \mathbf{R}^2$, define

$$J^+(S) = \bigcup_{x \in S} J^+(x).$$

(1.3) DEFINITION. For each ordinal number α, $X \in \mathscr{X}^r(\mathbf{R}^2)$ and $x \in \mathbf{R}^2$, call $J_\alpha(x)$ the *prolongational limit set of order* α, defined by transfinite induction, as follows,

1. $J_1(x) = J^+(x)$.
2. Suppose that for all $\beta < \alpha$, $J_\beta(x)$ is defined.

   (i) If α is a successor ordinal number, set

   $$J_\alpha(x) = J_1(J_{\alpha-1}(x)).$$

   (ii) If $\alpha \geq \omega$ is a limit ordinal number, where ω is the first transfinite number, set

$$J_\alpha(x) = \{y \in \mathbf{R}^2 : \exists x_n \to x, y_n \to y \text{ and ordinals } \beta_n < \alpha \text{ with } y_n \in J_{\beta_n}(x_n)\}.$$

It can be proved for any 2nd countable metric space that each prolongational limit set $J_\alpha(x)$ has a countable order α.

(1.4) DEFINITION. The *generalized recurrent set* $R(X)$ of X (also called Auslander recurrent set [1] or prolongational recurrent set) is defined by

$$R(X) = \{x \in \mathbf{R}^2 : x \in J_\alpha(x), \text{ for some ordinal } \alpha\}.$$

The nonwandering set $\Omega(X)$ is now the subset of $R(X)$ consisting of those points x for which $x \in J_1(x)$. (Note that we have $x \in J_1^+(x) \Leftrightarrow x \in J_1^-(x)$, which implies that all of $\Omega(X)$ is included.)

(1.5) DEFINITION. A point $p \in R(X)$, which is not a periodic point is called *prolongationally recurrent*.

To clarify the concepts of generalized recurrent set and prolongationally recurrent point some examples are given below. In the example (a), $R(X)$ is the set of all prolongationally recurrent points x plus the saddle node y. In (b) $R(X)$ is the whole closed shaded area. In (c) the three closed elliptic sectors constitute $R(X)$. In (d) we have an example in the sphere S^2 minus a point; with a sink in the other side of S^2. $R(X)$ is the set of the fixed saddles plus all separatrices that connect these saddles to infinity. All points of these separatrices are prolongationally recurrent. A similar example can be constructed for $x \in J_\alpha(x)$, α being any countable ordinal number.
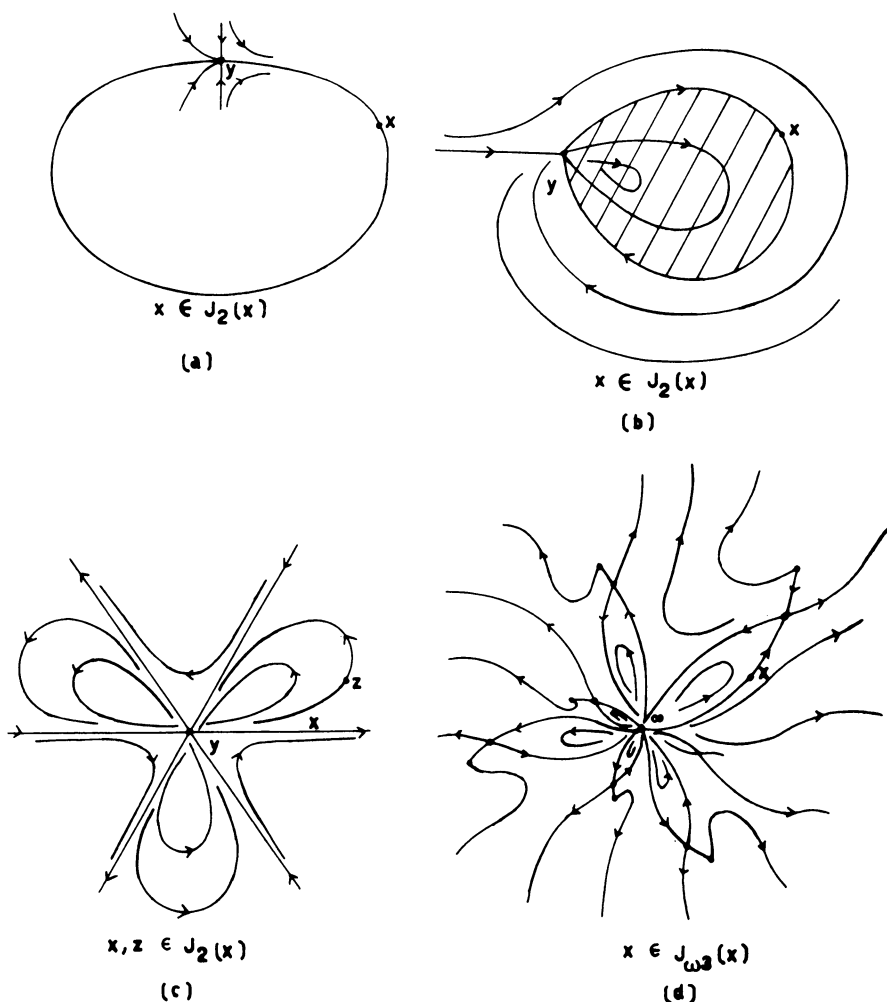
$x \in J_2(x)$

(a)

$x \in J_2(x)$

(b)

$x, z \in J_2(x)$

(c)

$x \in J_{\omega 3}(x)$

(d)

FIGURE (1.1)

(1.6) DEFINITION. An (ε, T)-*chain from* p *to* q is a finite collection of trajectory arcs or segments $[x_1, y_1], \ldots, [x_n, y_n]$, each of duration $\geq T$ such that $d(y_i, x_{i+1}) < \varepsilon$, $0 \leq i \leq n$, where $p = y_0$ and $q = x_{n+1}$.

(1.7) DEFINITION. A singularity y of X satisfies the *shadow property* if there exists a neighborhood N of y such that every $(\varepsilon, 1)$-chain in N can be δ-shadowed by a single trajectory arc of X and $\delta \to 0$ uniformly as $\varepsilon \to 0$.

An $(\varepsilon, 1)$-chain is said to be δ-shadowed by a single trajectory arc of X, if any point of the chain is less than δ distant from the trajectory arc and vice-versa. In [20] there is a description of all singularities which satisfy the shadow property, including the hyperbolic singularities.
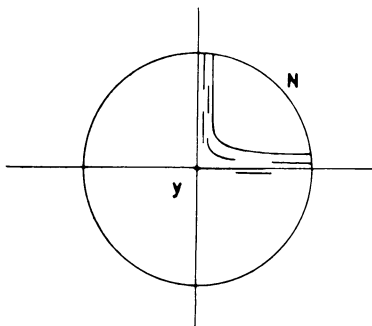
FIGURE (1.2)

(1.8) DEFINITION A singularity y of X is semihyperbolic if it is isolated and the Jacobian $J(X)_y = 0$, but its trace $\sigma(X)_y \neq 0$.

Semihyperbolic singularities are the saddle-node, the weak node or the weak saddle. It is possible to reduce their topological study by means of the restriction of X to the center manifold w^c_X, which is an invariant curve tangent at y to the null eigenvector of DX_y (see [3]). In the case of the saddle-node, assume the saddle-node y is $(0,0)$ and in local coordinates (x_1, x_2) around y the center manifold W^c_X is the x_1-axis. If $X = f(x_1, x_2)\partial/\partial x_1 + g(x_1, x_2)\partial/\partial x_2$, then the saddle-node is characterized by $g(x_1, x_2) = x_2 h(x_1, x_2)$, with $h(x_1, x_2) < 0$ and $f(x_1, 0) > 0$ or $f(x_1, 0) < 0$ for $x_1 \neq 0$.

We now state the theorem which is the object of this paper.

(1.9) THEOREM (CLOSING LEMMA). *Let p be a prolongationally recurrent point of $X \in \mathscr{X}^r(\mathbf{R}^2)$, $r \geq 1$, with singularities which are either semihyperbolic or satisfy the shadow property. Let $\mathscr{U}$ be a neighborhood of X in $\mathscr{X}^r(\mathbf{R}^2)$. Then there exists $Y \in \mathscr{U}$ such that Y has a closed orbit through p.*

It might be desirable to have such a Closing Lemma when the singularities are isolated and not too complicated. This does not seem to be an easy matter. In example (b) of Figure (1.1) the singularity does not satisfy our restriction and we do not know if we can get a closed orbit through $x \in R(X)$. In example (c) of Figure (1.1) the singularity also does not satisfy our requirements. Still the same method used in the proof of our theorem allows us to get a closed orbit through $x \in R(X)$. But through $z \in R(X)$, or through any point such that the corresponding α and ω-limit sets are the singularity y, this does not happen.

In §2 we show that if X is devoid of singularities on $\mathbf{R}^2$, the Closing Lemma is trivially true because $R(X)$ is empty. In §§3 and 4 we give a number of concepts and results necessary for our proof, of which the box-chain lemma is the central result. This lemma shows that when $y \in J_\alpha(x)$, α any ordinal, then there exists between x and y a chain of flowboxes and trajectory arcs connecting their central portions. The idea of the proof of our Closing Lemma is to consider a box-chain loop through a prolongationally recurrent point $p \in R(X)$. Next we perturb the vector field X locally in each flowbox in order to connect the two trajectory arcs of the chain that arrive and leave the flowbox. Then we come to the final step where the restriction on the nature of the singularities comes in. The semihyperbolic singularities existing in the chain can be eliminated through a perturbation, and

the ones which satisfy the shadow property can be bypassed. Thus we get our closed orbit.

We remark that in the plane the Closing Lemma on $\Omega(X)$ is true and trivial without any restriction on the nature of the singularities.

2. The nonsingular case. A vector field with no nonwandering points is not necessarily devoid of recurrence phenomena. The example in the cylinder $S^1 \times \mathbf{R}$ indicated in Figure (2.1) has a subtle kind of cycle, which belongs to the generalized recurrent set $R(X)$. Nevertheless on the plane, this fact does not occur. Actually we have the following.

(2.1) THEOREM. *Let X be a C^r vector field on $\mathbf{R}^2$. If $\Omega(X) = \varnothing$, then $R(X) = \varnothing$.*
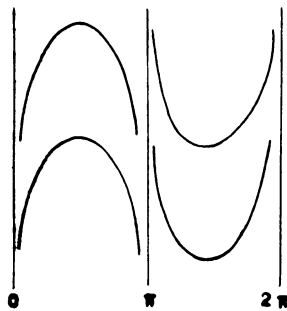

FIGURE (2.1)

The proof of this theorem is an easy consequence of a theorem of Nitecki [9] and the two easy lemmas below. First we give a definition.

(2.2) DEFINITION. Let φ be the X-flow. We call φ C^r nonexplosive, $r \geq 0$, if $\Omega(\varphi) = \varnothing$ and there exists a neighborhood $\mathcal{U}$ of φ in the Whitney C^r topology such that any flow $\psi \in \mathcal{U}$ has $\Omega(\psi) = \varnothing$.

The theorem of Nitecki gives several conditions equivalent to the property that a X-flow φ (on a noncompact manifold) and its C^0 perturbations have only wandering points, i.e., φ is C^0 nonexplosive. The equivalence we use here is

(2.3) φ is C^0 nonexplosive $\Leftrightarrow$ the generalized recurrent set $R(\varphi)$ is empty.

(2.4) LEMMA. *Let M be a noncompact (resp. compact) manifold and $\mathscr{X}_0^r(M)$ the set of all C^r nonsingular vector fields on M, $r \geq 0$. Then $\mathscr{X}_0^r(M)$ is an open set in $\mathscr{X}^r(M)$.*

PROOF. Consider $X_0 \in \mathscr{X}_0^r(M)$, so that X_0 is a section in the tangent bundle of M that does not intersect the 0-section. Then we can find a neighborhood N of the section X_0 in the Whitney topology (resp. topology of uniform convergence) such that any $Y \in N$ is nowhere zero, i.e., $Y \in \mathscr{X}_0^r(M)$. Thus $\mathscr{X}_0^r(M)$ is open in $\mathscr{X}^r(M)$. $\square$

(2.5) LEMMA. *X is nonsingular on $\mathbf{R}^2 \Leftrightarrow \Omega(X) = \varnothing$.*

PROOF. Suppose $\Omega(X) \neq \varnothing$. Let $p \in \Omega(X)$ and Σ be a transversal segment at p. Then for any neighborhood V of p, there exist $x, y \in V \cap \Sigma$ such that $y = \varphi_t(x)$

for some $t > 0$. Suppose y is the first point on the semitrajectory $\gamma_t(x)$ belonging to Σ. Consider the jordan curve L composed by the trajectory arc xy and the segment $[x, y] \subset \Sigma$. By the Poincaré-Bendixson theorem, in the interior of L there exists a singularity, which is a contradiction. Thus $\Omega(X) = \varnothing$.

Conversely, if $\Omega(X) = \varnothing$, it is obvious that X is nonsingular, and the lemma is proved. $\square$

PROOF OF THEOREM (2.1). Since X is a C^r vector field on $\mathbf{R}^2$, it follows from Lemmas (2.4) and (2.5) that if $\Omega(X) = \varnothing$ then the X-flow φ is C^r nonexplosive for any $r \geq 0$. In this case the theorem of Nitecki (2.3) implies that $R(X) = \varnothing$. $\square$

REMARK. Mendes proves in [7] the analogue of Theorem (2.1) for diffeomorphisms of $\mathbf{R}^2$. We can prove the flow case from the Mendes theorem for the time-one diffeomorphism φ_1 of the X-flow, but this is not immediate.

3. Flowbox system. Let X be a C^r-vector field on a noncompact manifold M^n, $n \geq 2$ endowed with a Riemannian metric. We denote the set of regular points of X by $M_0 \subset M^n$, and the X-flow by φ.

Introduce a continuous system of flowboxes, associating a flowbox N_x to each point $x \in M_0$ in a continuous way. We need in §4 the fact that if two flowboxes N_x and N_y of the system intersect each other, their union contains a common transversal, that is, a segment transversal to X, which crosses all trajectories of X in N_x and in N_y. For this we define on an orientable manifold M^2 what we call the orthogonal system of flowboxes.

Consider $T_x M_0 = \text{Span}(X) \oplus \text{Span}(X^\perp)$, where $X^\perp$ is a vector field on M^2 orthogonal to X with flow $\varphi^\perp$. For any $x \in M_0$, there is a $\delta > 0$ and a neighborhood V of x, $V \subset M_0$ such that for all $|t_0| < \delta$, $|\tau_0| < \delta$ the intersection of the trajectory arcs $\{\varphi_t(\varphi_{\tau_0}^\perp(x))\}$ and $\{\varphi_\tau^\perp(\varphi_{t_0}(x))\}$ is a unique point in V, and the map $f_{x,\delta}: [-\delta, \delta] \times [-\delta, \delta] \to M_0$, defined by $f_{x,\delta}(t_0, \tau_0) = \{\varphi_t(\varphi_{\tau_0}^\perp(x))\} \cap \{\varphi_\tau^\perp(\varphi_{t_0}(x))\}$ is a C^r-embedding. Since M^2 is locally compact we may define a positive continuous function $\delta(x)$ on M_0, such that $f_{x,\delta(x)}$ is an embedding for each $x \in M_0$.

(3.1) DEFINITION. An *orthogonal system of flowboxes* is the set $\mathscr{F} = \{f_{x,\delta(x)}\}$, where $\delta(x)$ is a positive continuous function on M_0, chosen as above.

(3.2) DEFINITION. The *flowbox of radius* $\delta(x)$ at $x \in M_0$ is the image N_x of the map $f_{x,\delta(x)}$.

(3.3) DEFINITION. If $0 < c < 1$, the *c-central portion* of the interval $[-\delta, \delta]$ is the interval $[-c\delta, c\delta]$ and the c-central portion of N_x is the set

$$\{\{\varphi_t(\varphi_{\tau_0}^\perp(x))\} \cap \{\varphi_\tau^\perp(\varphi_{t_0}(x))\}: |t_0| \leq \delta(x), |\tau_0| \leq c\delta(x)\}.$$

For more general situations below, we define a canonical system of flowboxes in a manifold M of arbitrary dimension as follows. For each $x \in M_0$, there exists a $\delta > 0$ giving the C^r-embedding $f_{x,\delta}: [-\delta, \delta] \times \pi_x(\delta) \to M_0$, defined by $f_{x,\delta}(t, v) = \varphi_t(\exp_x(v))$, where $\pi_x(\delta)$ is the disc of radius δ in the hyperplane complementary to X in $T_x M_0$. As noted above, we may choose a positive function δ on M_0 so that $f_{x,\delta(x)}$ is an embedding for each $x \in M_0$.

(3.4) DEFINITION. Call $\mathscr{F} = \{f_{x,\delta}\}$ a *canonical system of flowboxes*. Just as in the orthogonal system of flowboxes, we define the flowboxes N_x of radius $\delta(x)$ at $x \in M_0$ and the c-central portion of N_x for a canonical system of flowboxes.

By (3.1) and (3.2), if two flowboxes N_x and N_y in an orthogonal system of flowboxes intersect each other, their union has a common transversal, indeed a

trajectory arc of the vector field $X^\perp$. This does not necessarily happen for a canonical system of flowboxes.

Fix an orthogonal system of flowboxes on $M_0 \subset M^2$.

(3.5) LIFT LEMMA. *Given an $\varepsilon(x)$-neighborhood $\mathscr{U}$ of $X \in x^r(M^2)$, $r \geq 1$, there exists a continuous function $0 < c(x) < 1$ on M_0 such that any two points p, q in opposite transversal edges of the $c(x)$-central portion of the flowbox N_x around x can be connected by a trajectory arc of a vector field $Y \in \mathscr{U}$, with $Y \equiv X$ outside N_x.*
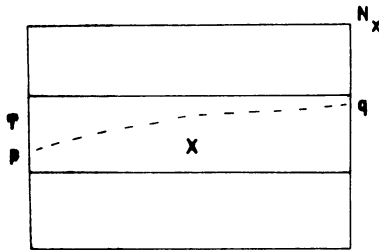

FIGURE (3.1)

PROOF. By (3.1) and (3.2), each flowbox N_x has radius $\delta(x)$, where $\delta(x)$ is a positive continuous function on M_0. We can take $\delta(x) < 1$. For $0 \leq u \leq 1$, define the vector field $X(u)$ on M^2 by $X(u) = X + (\varepsilon/B)u\beta Z$, where $\varepsilon \leq \inf_{z \in N_x} \varepsilon(z)$, β is a bump function such that $0 < \beta \leq 1$ inside N_x and $\beta \equiv 0$ outside N_x, $Z = X^\perp$ normalized and $B = \max_{0 \leq j \leq r} \sup\{\|\beta^{(j)}(z)\|; z \in N_x\}$. It follows that $X(u) \in \mathscr{U}$ and $X(u) \equiv X$ outside N_x.

We use the idea of a "minimum lift" from [12], as follows. Consider a compact central portion of N_x and let τ denote the segment determined by this central portion on the left transversal edge of N_x. (See Figure (3.1).) For every point $y \in \tau$, consider the trajectories of $X(0)$ and $X(1)$ through y and call $l(y) > 0$ the length of the segment determined by them on the right transversal edge of N_x. Since $l(y)$ is continuous in y and τ is compact, there is a minimum lift $l > 0$ such that $l(y) \geq l$ for all $y \in \tau$. Clearly if we diminish τ, the minimum lift increases. Thus we can choose c, $0 < c < 1$, so small that the minimum lift corresponding to the c-central portion of N_x is greater than the radius $c\delta$ of this central portion. Thus if p and q are points in opposite transversal edges of the c-central portion of N_x, there exists a vector field $Y = X(u_0)$ for some $0 \leq u_0 \leq 1$, connecting p to q by a trajectory are through N_x and $Y \in \mathscr{U}(X)$. By the continuity of the orthogonal system of flowboxes we can choose the function c continuous in x, and the lemma is proved. $\square$

4. Box-chains. Let $X \in \mathscr{X}^r(M)$ be a vector field with isolated singularities. Because some results are valid in any dimension, we begin in M^n, a manifold of arbitrary dimension. Fix a canonical covering of $M_0 \subset M$ by flowboxes of radius $\delta(x)$, as in (3.4).

(4.1) DEFINITION. A *box-chain* from x to y is a finite and alternate sequence of flowboxes N_i and curves γ_i, such that γ_i is either a trajectory arc or two trajectory arcs with a singularity between them and γ_i joins N_i to N_{i+1}, $i = 1, 2, \ldots, n - 1$.
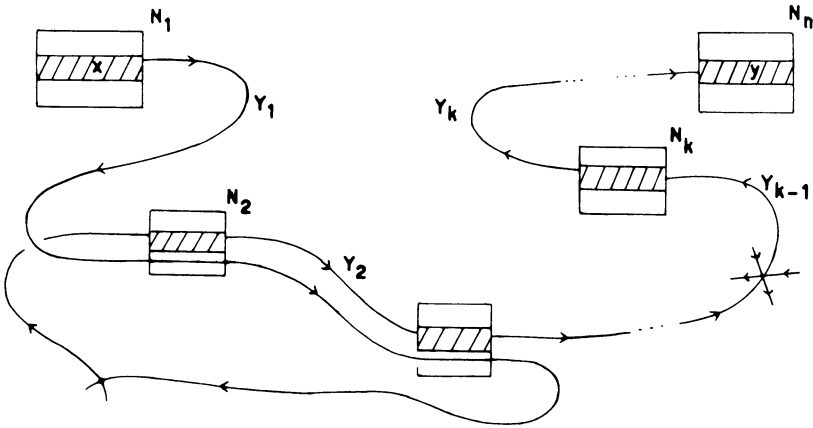
FIGURE (4.1)

(4.2) DEFINITION. If $c = c(x)$ is a positive function on M_0, the box-chain is *c-central* if γ_i joins the *c*-central portion of N_i to that of N_{i+1}. See Figure (4.1).

(4.3) BOX-CHAIN LEMMA. *Let $y \in J_\alpha(x)$, where α is any ordinal number and x, y are any points in M_0. Assume the X-flow has only isolated singularities. Then, for every continuous function $c = c(x)$ on M_0, $0 < c < 1$, there exists a c-central box-chain from x to y.*

PROOF. We prove this lemma by transfinite induction on α.

STEP 1. Take $\alpha = k > 0$, an integer, so

$$y \in J_k(x) = \underbrace{J_1(J_1 \cdots (J_1(x)) \cdots)}_{k\text{-times}}.$$

This implies we can choose points $p_0, p_1, \ldots, p_m$, $m \geq k$, with $p_0 = x$ and $p_m = y$ such that $p_{i+1} \in J_1(p_i)$ and some p_i may be singularities. We have $m \geq k$ because we assume, without loss of generality, that for each singularity p_j we choose regular points p_{j-1} and p_{j+1} in one of its α-separatrix and ω-separatrix respectively such that $p_{j-1} \in J_1(p_{j-2})$ and $p_{j+2} \in J_1(p_{j+1})$.

Let $z_0 = p_0, z_1, \ldots, z_n = y$ be the regular points p_i. In the canonical system of flowboxes, there exist flowboxes N_i around each z_i, $i = 1, \ldots, n$. We can have $z_{i+1} \in J_1(z_i)$ or $p_j \in J_1(z_i)$ and $z_{i+1} \in J_1(p_j)$, p_j a singularity. In any case there exist sequences $z_i^n \to z_i$ and $t_n^i \to \infty$ such that $\varphi_{t_n^i}(z_i^n) \to z_{i+1}$ or $\varphi_{t_n^i}(z_i^n) \to p_j$. So for each i, there exists a curve γ_i which is either a trajectory arc of $\mathcal{O}_+(z_i^n)$ for some n or two trajectory arcs of $\mathcal{O}_+(z_i)$ and $\mathcal{O}_-(z_{i+1})$ with a singularity p_j at their intersection connecting the central portions of N_i and N_{i+1}. In the first case we can choose n large enough so γ_i joins the *c*-central portion of N_i to that of N_{i+1}, for any $0 < c < 1$, by the definition of prolongational limit set. In the second case, γ_i always leaves N_i and arrives at N_{i+1} in their *c*-central portion, for any c, since γ_i is composed of trajectory arcs through z_i and z_{i+1}. Then we have a *c*-central box-chain from x to y.

STEP 2. Take α to be any ordinal number.

We suppose the lemma is not true for a subset S of the ordinal numbers. From the well-ordering of S, there exists a smallest element $\bar{\alpha}$. Since the lemma is true for all $\beta < \bar{\alpha}$, let us prove it is true as well for $\bar{\alpha}$, which is absurd. We have two cases

(a) Suppose $\bar{\alpha}$ is a limit ordinal number. By Definition (1.3),

$$J_{\bar{\alpha}}(x) = \{y \mid \exists x_n \to x, y_n \to y \text{ such that } y_n \in J_{\beta_n}(x_n), \text{ where } \beta_n < \bar{\alpha}\}.$$

Since $\beta_n < \bar{\alpha}$, by the induction hypothesis the lemma is true for $y_n \in J_{\beta_n}(x_n)$, i.e., there exists a c-central box-chain from x_n to y_n for any $0 < c < 1$. Thus we can choose n so large that x_n and y_n are in the c-central portion of N_x and N_y respectively and the $c/2$-central portion of N_{x_n} and N_{y_n} are in the c-central portion of N_x and N_y respectively. This implies that the trajectory arc γ_0 leaves the c-central portion of N_x and γ_{n-1} arrives at that of N_y. Thus we have a c-central box-chain from x to y, where $y \in J_{\bar{\alpha}}(x)$, which is a contradiction.

(b) Suppose $\bar{\alpha}$ is not a limit ordinal number. We have $y \in J_{\bar{\alpha}}(x) = J_1(J_{\bar{\alpha}-1}(x))$. This implies that there exists $y' \in J_{\bar{\alpha}-1}(y)$ such that $y \in J_1(y')$. Since $\bar{\alpha} - 1 < \bar{\alpha}$, by the induction hypothesis there exists a c-central box-chain from x to y', for any $0 < c < 1$. As $y \in J_1(y')$, by definition we can choose a trajectory arc γ_n leaving the c-central portion of $N_{y'}$ and arriving at that of N_y, for any $0 < c < 1$. Then, adding the trajectory arc γ_n and the flowbox N_y to the box-chain from x to y', obtained above, we get a c-central box-chain from x to y, for any $0 < c < 1$, which is a contradiction. Thus (4.3) is true for all ordinal numbers α and $x, y \in M_0$, such that $y \in J_\alpha(x)$. □

(4.4) THEOREM. *Let X be a vector field on M as in (4.3) and $x \in M_0$ be prolongationally recurrent, i.e., $x \in J_\alpha(x)$ for some ordinal α. Then for every continuous function c and M_0, $0 < c < 1$, there exists a c-central box-chain loop from x to x.*

PROOF. It follows immediately from the box-chain lemma, where we consider $y = x$.

(4.5) BOX-CHAIN SIMPLE IN $\mathbf{R}^2$. From the preceding theorem we get a c-central box-chain from x to x, where x is a prolongationally recurrent point. But that chain might cross some of the flowboxes twice or more times, producing loops, and it might have flowboxes intersecting each other. In order to get a closed orbit from a c-central box-chain, we show here that in the plane we can make it box-chain simple, in the sense that the curve γ_i crosses N_i just once, leaving N_i and arriving at N_{i+1}, without crossing any N_j, $j \neq i, i+1$ and that the flowboxes N_i are disjoint. Here we consider the orthogonal system flowboxes as in (3.1). Using the theorem (4.4), consider a closed continuous curve Γ through the point x, composed by arcs so that

$$\Gamma = x, \hat{\tau}_0, \gamma_0, \tau_1, \gamma_1, \tau_2, \ldots, \gamma_{n-2}, \tau_{n-1}, \gamma_{n-1}, \hat{\tau}_n, x$$

where $\gamma_0, \gamma_1, \ldots, \gamma_{n-1}$ are the trajectory arcs given by the box-chain and τ_i is a transversal segment inside the flowbox N_i connecting the curve γ_{i-1} to γ_i while $\hat{\tau}_0, \hat{\tau}_n$ are the two parts of the transversal segment, which contain x, as shown in Figure (4.2). Here Γ can cross itself only in the transversal segments τ_i, or in the singularities since the curves γ_i are trajectory arcs. Thus let us choose first a simplified oriented loop Γ' from x to x, i.e., each γ_i in Γ' crosses N_i and N_{i+1} just

once and γ_i does not cross any N_j, $j \neq i$, $i+1$. We do that choosing the most future arc, when Γ crosses itself, as in Figure (4.3). Roughly speaking, we call Γ' a box-chain loop through x, simplified through the construction of the oriented loop Γ' above.

We know that a curve crosses a transversal segment τ just once in the topological sense if it crosses the transversal segment as in Figure 4.4.

(4.6) LEMMA. *A simplified box-chain loop in* $\mathbf{R}^2$ *crosses a transversal segment at most once in the topological sense.*

PROOF. Suppose a simplified box-chain loop Γ' crosses a transversal segment τ twice. We obtain a jordan curve L composed by the part of Γ' between the two crossings plus the transversal segment connecting them. (See Figure 4.5.)

By the Jordan curve theorem, Γ' enters in a disc D limited by L and cannot leave it, since Γ' does not cross itself (Γ' is simplified) and cannot cross the transversal segment τ in the opposite direction. This is absurd, since Γ' is a loop. □
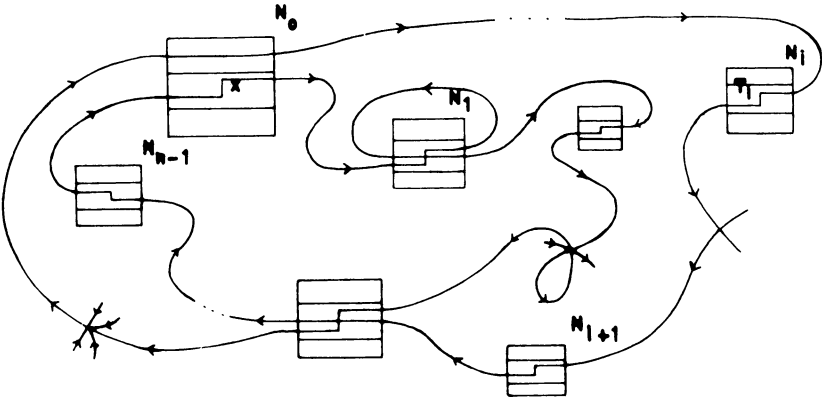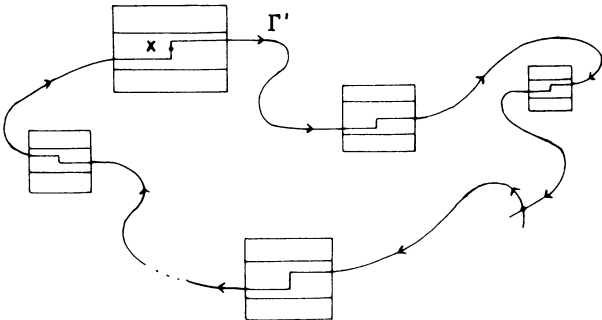

FIGURE (4.2)

FIGURE (4.3)

FIGURE (4.4)

(4.7) LEMMA. *Any simplified box-chain loop Γ' in $\mathbf{R}^2$ has these properties.*
(a) *No loop arc γ_i meets a flowbox N_j, $j \neq i$, $i + 1$.*
(b) *The flowboxes of the box-chain Γ' are disjoint.*

PROOF. (a) In fact, since the arc $\gamma_{j-1}\gamma_j$ already crosses the transversal segment τ_j through $z_j \in N_j$, by the Lemma (4.6) it is not possible to have γ_i crossing N_j, for any $j \neq i$, $i + 1$.

(b) Suppose $N_i \cap N_j \neq \varnothing$, $i \neq j$. From the orthogonal system of flowbox, any flowbox around a regular point has transversal segments that are trajectory arcs of the orthogonal vector field $X^\perp$. Thus $N_i \cup N_j$ has a common transversal segment τ, as seen in Figure 4.6. By Lemma (4.6), intersection is impossible, or Γ' would cross the transversal segment twice. So $N_i \cap N_j = \varnothing$ for any i, j, $i \neq j$. $\square$

By Lemma (4.7), the box-chain loop Γ' is in fact simple, and by construction it is even more central with respect to its flowboxes than Γ, and the flowboxes in Γ' might be less in number than the ones in Γ. See Figures (4.2) and (4.3).

REMARK 1. Γ' might contain some singularities, but if we assume X has only hyperbolic singularities (in this case saddles), then we can get a nonsingular c-central chain-box loop directly from the box-chain lemma. In fact, in the box-chain lemma proof, when constructing the box-chain, we can avoid taking singularities in the choice of the points prolongationally related. This is possible because the singularities are hyperbolic. Thus we can always choose in $J_1(z_i)$, for some point z_i in the α-separatrix of a singularity p_j, a regular point in the ω-separatrix of this singularity, instead of the singularity itself. We cannot always do this when the singularities are isolated as in Figure (4.7)(b), where we cannot avoid p_j.

We need two more steps to obtain a closed orbit through $x \in R(X)$ proving Closing Lemma (1.9). The first step is to prove the Singular Closing Lemma, which gives a closed curve through x, composed of trajectory arcs and singularities between them. The second step is to perturb the vector field obtained to eliminate
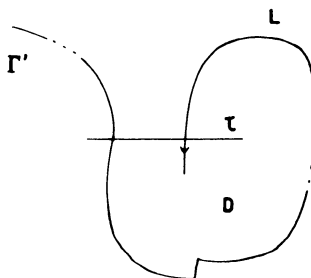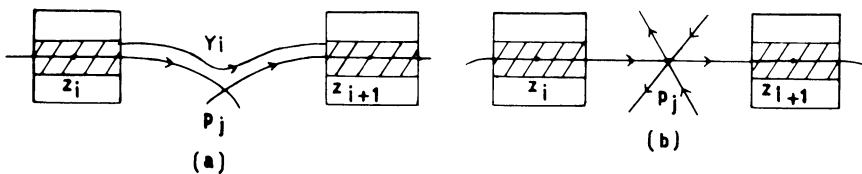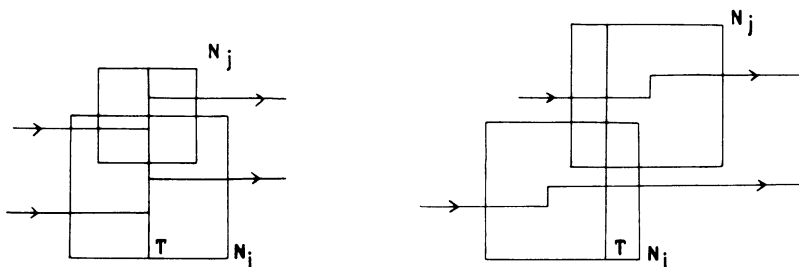

FIGURE (4.5)

FIGURE (4.6)

FIGURE (4.7)

the singularities of the closed curve, getting a closed orbit through x. We need first a simple lemma.

(4.8) LEMMA. *Suppose* $X \in \mathscr{X}^r(\mathbf{R}^2)$, $r \geq 1$ *and* $\mathscr{U}$ *is a* $\varepsilon(x)$-*neighborhood of* X *in* $\mathscr{X}^r(\mathbf{R}^2)$. *If* $\Delta_1, \Delta_2, \ldots$ *in* $\mathscr{X}^r(\mathbf{R}^2)$ *have disjoint supports and* $Y_i = X + \Delta_i$ *is in* $\mathscr{U}$ *for each* i, *then* $Y = X + \sum_i \Delta_i$ *is also in* $\mathscr{U}$.

PROOF. The $\varepsilon(x)$-neighborhood $\mathscr{U}$ of X in the Whitney topology is defined pointwise by

$$\mathscr{U}(X, \varepsilon(x)) = \{Y \in \mathscr{X}^r(\mathbf{R}^2) : \|Y(x) - X(x)\|_r < \varepsilon(x) \text{ for all } x \in \mathbf{R}^2\},$$

where

$$\|Y(x) - X(x)\|_r = |Y(x) - X(x)| + \|D^1(Y - X)(x)\| + \cdots + \|D^r(Y - X)(x)\|.$$

Since each $Y_i = X + \Delta_i \in \mathscr{U}$ and $\Delta_1, \Delta_2, \ldots$ have disjoint supports, then

$$\|Y(x) - X(x)\|_r = \left\| \sum_i \Delta_i \right\|_r$$

$$= |\Delta_1(x) + \Delta_2(x) + \cdots| + \cdots + \|D^r(\Delta_1)(x) + D^r(\Delta_2)(x) + \cdots\| \leq \varepsilon(x),$$

which implies $Y \in \mathscr{U}$.

(4.9) SINGULAR CLOSING LEMMA. *Suppose* $X \in \mathscr{X}^r(\mathbf{R}^2)$ *with all singularities isolated,* $p \in R(X)$ *a prolongationally recurrent point and* $\mathscr{U}$ *a neighborhood of* X *in* $\mathscr{X}^r(\mathbf{R}^2)$. *Then there exists a vector field* $Z \in \mathscr{U}$ *with a closed curve through* p, *composed of trajectory arcs and singularities of* Z.

PROOF. By (4.4) and (4.5), for every continuous function $0 < c < 1$ on $\mathbf{R}_0^2 \subset \mathbf{R}^2$, $\mathbf{R}_0^2$ being the set of regular points of X, there exists a c-central box chain loop Γ' from p to itself, which is box simple. Consider a smaller neighborhood $\mathscr{U}'$ of X, $\mathscr{U}' \subset \mathscr{U}$. By (3.5) there is a positive continuous function $c < 1$ on $\mathbf{R}_0^2$ such that

any two points in opposite transversal edges of a c-central portion of a flowbox N_x can be connected by a X'-trajectory across N_x, where $X' \in \mathscr{U}'$ is such that $\mathrm{supp}(X' - X) \subset N_x$. Consider the flowboxes N_i, $i = 1, \ldots, n$, of the box-chain and make this connection construction in each N_i for the curves γ_{i-1} and γ_i, that arrive and leave N_i respectively in opposite transversal edges of the c-central portion of N_i. Thus we get $Y_1, Y_2, \ldots, Y_n \in \mathscr{U}'$. By Lemma (4.8), $Y = X = \sum_{i=1}^{\infty}(Y_i - X)$ also belongs to $\mathscr{U}'$. By construction, Y has a closed curve γ composed only of trajectory arcs and singularities between them, which may not contain p but passes near p. Let q be a point of λ close to p. We consider a local translation $f: \mathbf{R}^2 \to \mathbf{R}^2$ that takes q to p, which is a C^∞ diffeomorphism near the identity in a small neighborhood V of p and q and equal to the identity outside a neighborhood containing $\overline{V}$. This f induces a map f_* (see [11]) such that the vector field $Z = f_* Y = df \circ Y \circ f^{-1}$ has a closed curve $\bar{\lambda}$ through p, which is a translation of λ and $z \in \mathscr{U}$. Then (4.9) is proved. $\square$

REMARK 2. Although it is intuitive that such a local translation f exists, we give a construction. Consider a C^∞ function ψ on $\mathbf{R}^2$ such that $\mathrm{supp}\, \psi = D_1^2$ and $\psi = 1$ on $D_{1/2}^2$. Consider the vector field $W = W(z; q, v, \mu) = \psi((z - q)/\mu)v$ on $\mathbf{R}^2$ with $z, q, v \in \mathbf{R}^2$, q given above, and $\mu > 0$ the radius of a disc with center at q. Then W is zero for z outside the disc D_μ^2 and W is equal v for z inside $D_{\mu/2}^2$. Moreover $W \to 0$ when $v \to 0$. Define the diffeomorphism $f: \mathbf{R}^2 \to \mathbf{R}^2$ for given v and μ by $f(z; q, v, \mu) = \varphi_1(z; W(\cdot, q, v, \mu))$, where φ_1 is the time one diffeomorphism of the W-flow. Since $W \to 0$ when $v \to 0$, f is near the identity inside $\Gamma_{\mu/2}^2$ for small v and f is the identity outside D_μ^2 because W is zero there. Since $f(q) = q + v$, taking $v = p - q$ gives f all properties required in the proof of (4.9).

REMARK 3. If the singularities of X are hyperbolic, we have the Closing Lemma. In fact, by Remark 1 we can get a nonsingular box-chain through p, directly from the Box-chain Lemma and make it simple by (4.5). By the same arguments used in the proof of (4.9), we can get a closed orbit through p for a vector field arbitrarily close to X.

We now remove the singularities of the closed curve given by the Singular Closing Lemma to obtain our main result.

CLOSING LEMMA. *Suppose the singularities of $X \in \mathscr{X}^r(\mathbf{R}^2)$ are all semihyperbolic or satisfy the shadow property. Given a prolongationally recurrent point $p \in R(X)$, and a neighborhood $\mathscr{U}$ of X in $\mathscr{X}^r(\mathbf{R}^2)$, there exists $Z \in \mathscr{U}$ with a closed orbit through p.*

PROOF. We choose a smaller neighborhood $\mathscr{U}' \subset \mathscr{U}$ of X, and by the Singular Closing Lemma there exists a vector field $Y \in \mathscr{U}'$ such that Y has a closed curve λ through p, composed of trajectory arcs and singularities p_i satisfying the shadow property (1.7) and semihyperbolic singularities q_j, $i = 1, \ldots, m$, $j = 1, \ldots, n$. We consider a neighborhood $\mathscr{U}_0$ of Y, $\mathscr{U}_0 \subset \mathscr{U}$.

Since each p_i satisfies the shadow property, by (1.7) there exists a neighborhood W_i of p_i such that every $(\varepsilon, 1)$-chain in W_i can be δ-shadowed by a single trajectory arc and $\delta \to 0$ uniformly as $\varepsilon \to 0$. Thus we can have a $(\varepsilon, 1)$-chain $[a_i, b_i]$, $[c_i, d_i]$ with the two trajectory arcs of λ contained in $\overline{W}_i$, which are respectively the trajectory arc that goes to p_i and the trajectory arc that leaves p_i. We take a_i,
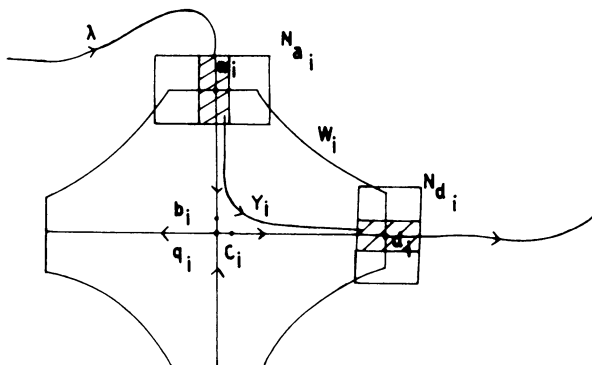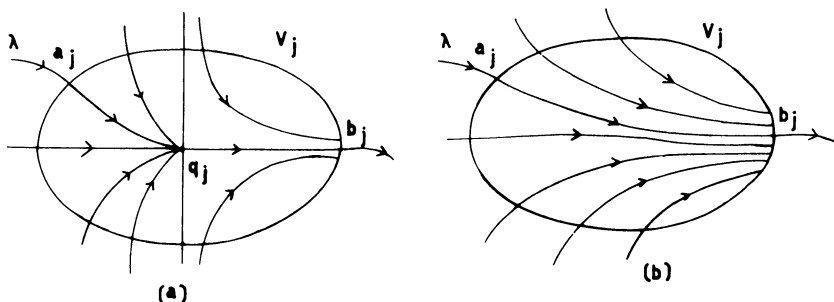
FIGURE (4.8)

FIGURE (4.9)

d_i as the intersection points of λ with ∂W_i. By the shadow property, this chain is
δ-shadowed by a single trajectory arc γ_i of Y. See Figure (4.8).

Consider disjoint flowboxes N_{a_i} and N_{d_i} around a_i and d_i, for each i. By (3.5)
we have a c-central portion in N_{a_i} and in N_{d_i} such that any two points in opposite
transversal edges of the c-central portion of N_{a_i} and of N_{d_i} can be connected by a
trajectory arc of Z_i and of Z_i', $Z_i, Z_i' \in \mathcal{U}_0$, with $Z_i \equiv Y$ and $Z_i' \equiv Y$ outside N_{a_i} and
N_{d_i} respectively. Since $\delta \to 0$, uniformly as $\varepsilon \to 0$, we can have the trajectory arc
γ_i leaving ∂N_{a_i} and arriving at ∂N_{d_i} in their c-central portions. Next we connect
λ to γ_i by a Z_i-trajectory across N_{a_i} and γ_i to λ by a Z_i'-trajectory across N_{d_i}.
Thus we remove p_i from λ, for each i.

Clearly the semihyperbolic singularities of Y that could belong to the box-chain
loop Γ' which starts the closed curve λ are saddle-nodes and weak saddles. The
weak saddles satisfy the shadow property and they are removed as above. Then
the singularities q_j to be removed are all saddle-nodes.

Choose a small neighborhood V_j of each q_j such that the only singularity of Y
in V_j is q_j, while $V_1, \ldots, V_n, N_{a_1}, \ldots, N_{a_n}, N_{d_1}, \ldots, N_{d_n}$ are disjoint and in V_j we
can write

$$Y = f(x_1, x_2) \frac{\partial}{\partial x_1} + g(x_1, x_2) \frac{\partial}{\partial x_2},$$

where the saddle-node $q_j(0,0)$. By the remarks following (1.8) we have $g(x_1, 0) \equiv 0$
and $f(x_1, 0) > 0$ or $f(x_1, 0) < 0$ for $x_1 \neq 0$. We assume $f(x_1, 0) > 0$. The closed

curve λ comes in V_j by a trajectory arc crossing the boundary ∂V_j at a_j and leaves V_j by the unstable separatrix at b_j. (See Figure (4.9)(a).)

Suppose φ_j is a C^∞ function with $0 < \varphi_j(x_1, x_2) \le 1$ inside V_j and $\varphi_j(x_1, x_2) \equiv 0$ outside V_j. Then for arbitrarily small values of $\varepsilon_j > 0$ and of appropriately chosen μ_j, the family

$$Y_j = Y + \varepsilon_j \varphi_j \frac{\partial}{\partial x_1} + \mu_j \varphi_j \frac{\partial}{\partial x_2}$$

has the following properties, based on the continuity of integral curves with respect to parameters.

(1) $Y_j \in \mathscr{U}_0$,
(2) $Y_j \equiv Y$ off V_j,
(3) q_j is removed and $Y_j(x_1, x_2) \ne 0$ for $(x_1, x_2) \in V_j$,
(4) the trajectory arc of Y_j through a_j passes through b_j. (See Figure (4.9)(b).)

Since $N_{a_1}, N_{a_2}, \ldots, N_{a_m}$, $N_{d_1}, N_{d_2}, \ldots, N_{d_m}$, $V_1, V_2, \ldots, V_n$ are disjoint and $\text{supp}(Z_i - Y) = N_{a_i}$, $\text{supp}(Z'_i - Y) = N_{d_i}$, and $\text{supp}(Y_j - Y) = V_j$, (4.8) shows that

$$Z = Y + \sum(Z_i - Y) + \sum(Z'_i - Y) + \sum(Y_j - Y), \qquad i = 1, \ldots, m, \ j = 1, \ldots, n,$$

lies in $\mathscr{U}_0 \subset \mathscr{U}$ and by construction Z has a closed orbit through p. $\square$

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